# BAYESIAN GAMES ON A MAXMIN NETWORK ROUTER

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## ABSTRACT
In this paper, we consider network games with incomplete information. In particular, we apply a game-theoretic network model to analyze Bayesian games on a MaxMin router. The results show that the MaxMin mechanism induces desirable Nash equilibria.

## Categories and Subject Descriptors
F.2.2 **[Theory of Computation]**: Analysis of Algorithms and Problem Complexity—Nonnumerical Algorithms and Problems; G.2.2 **[Mathematics of Computing]**: Discrete Mathematics—graph theory, network problems

## General Terms
Algorithms, Theory

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Bayesian games, MaxMin fairness, Network games, Nash equilibrium, Algorithmic Game Theory

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## 1. INTRODUCTION
The interaction of independent Internet flows that compete for common network resources like the bandwidth at routers has recently been addressed with tools from algorithmic game theory. In this context, the flows are considered independent players who seek to optimize personal utility functions, such as the bandwidth. The mechanism of the game is determined by the network infrastructure and the policies used. The solution concept commonly used is the Nash equilibrium (NE), i.e., a state of the game in which no player has anything to gain by unilaterally changing her strategy.

An assumption underlying Nash equilibria is that each player holds the correct belief about the other players' actions. To do so, a player must know the game she is playing. However, in many situations the participants are not perfectly informed about their opponents' characteristics: firms may not know each others' cost functions, players-flows competing for bandwidth of a network may not know the exact number of active flows etc. The model of a Bayesian game [4], generalizes the notion of a strategic game to cover situations in which each player is imperfectly informed about some aspect of her environment relevant to her choice of an action. The players may be uncertain about the exact state and instead assign a probability distribution over the possible states

The model that we will use, the Window-game (presented in [2]), has a router and N flows, and is played synchronously, in one or more rounds. Every flow is a player that selects in each round the size of its congestion window. The congestion window of a TCP flow is a parameter that determines the maximum number of packets that the flow is allowed to have outstanding in the network at each moment [1]. Real TCP flows adjust their congestion window to control their transmission rate. The router of the Window-game (the mechanism of the game) receives the actions of all the flows and decides how the capacity is allocated. The utility or payoff P(i) of each flow i is equal to the capacity that it obtains (transmitted packets) from the router in each round minus a cost g for each dropped packet, that is:

$$P_i = \text{transmitted}_i - g \cdot \text{dropped}_i \qquad (1)$$

We assume that the cost g for each dropped packet is constant for a certain game and is the same for all players. The general form of the games we study is a Window-game with a router of capacity C and N players. The game is played in one round and each player i chooses its window size $w_i \leq C$. The router policy is MaxMin. The cost for a lost packet is g. The number of the active players (players that can submit packets) of each round is a random variable n. The uncertainty stems from the fact that players-flows may not know the total number of players that are active in each round but only a corresponding probability distribution. Let w=C/n be the Fair Share (FS) of each player when n active players compete for a capacity C. In case of overflow, a MaxMin fair router satisfies any request that does not exceed FS and splits the remaining router capacity (if any) to flows with larger windows. The complete information version of this game has been studied in [2].

In practice, real TCP flows are actually playing a repeated game, which means that the flow will, in most cases, have an estimation of its fair share from the previous rounds. The probability distribution over the possible states of the game is a way of adding this partial information to our model.

We first analyze the NE of some simple toy-case scenarios and then proceed to the main result of this work; an expression for symmetric NE of an interesting class of symmetric Bayesian Window-games. For simplicity, we will assume in this work that all FS values are integers.

## 2. TOY CASES OF NON-SYMMETRIC GAMES

We present some toy-cases of non-symmetric Bayesian games on a MaxMin router.

**Game 1.** Two players A and X and router capacity C=12. Player A is always active, while player X receives a signal that informs her weather she is active or not. The probability of each case is 1/2. While both players know all the above information, the contents of X' signal is not revealed to A.

**Solution sketch.** All strategies with w<FS=6 are strictly dominated by the strategy w=FS and, thus, can be excluded. For g=0, all the remaining strategies (6≤w≤12) give the same constant payoff P(w)=6 for both players, since the whole capacity is used and they both transmit a window size equal to FS=6, with zero cost for the lost packets (if w>6). By definition, player X can be active or not. When we refer to player X without any other specification, we will implicitly refer to the active type of X. The same holds for the rest of this work. For g>0, player X knows that A will play at least the FS. Thus, a capacity equal to FS remains available for X. There is no incentive for X to leave the w=6 strategy, since a greater w will only reduce her payoff due to the cost of lost packets. Player's A payoff is equal to the sum of the products of her payoff at each of the two possible states with the probability of each state. It can be shown that these are the NE ($w_A$,$w_X$) of the game:

**Table 1. The NE of Game 1**

| g | g=0 | 0<g<1 | g=1 | g>1 |
|---|---|---|---|---|
| NE | $(12,x)\,\forall x\geq 6$ | $(12,6)$ | $(x,6)\,\forall x\geq 6$ | $(6,6)$ |

**Game 2.** Same as Game 1, but with an additional player B, who, just like A, is always active.

**Solution sketch**. With an analysis similar to Game 1, it can be shown that these are the NE ($w_A$, $w_B$, $w_X$) of the game:

**Table 2. The NE of Game 2**

| g | g=0 | 0<g<1 | g=1 | g>1 |
|---|---|---|---|---|
| NE | $(x,y,z)$, $\forall x,y\geq 6$, $\forall z\geq 4$ | (6,6,4) | (x,y,4) $\forall x,y\in[4,6]$ | (4,4,4) |

**Comments**. It is clear that with MaxMin, more information leads generally to NE with better outcomes for players with this information. In many similar scenarios, the choice of g>1 generally gives fair NE.

## 3. A GENERAL SYMMETRIC (GS) GAME

**Game GS1.** Game with N players in which the number of active players is uniformly distributed in {1, 2, …, N}. In this game, each player receives a signal that informs her if she is active. Every active player assigns equal probability (p=1/N) to each of the N possible game states, where 0,1,2…, or N-1other players are active, respectively.

For each of these N different cases of the Bayesian game, there is a different FS. For example if N=5 and C=60:

**Table 3. The possible FS values of Game GS1**

| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| FS | 60 | 30 | 20 | 15 | 12 |

Due to the symmetric nature of the game, we will focus on its symmetric NE. Thus, we assume symmetric profiles for the game, i.e., all active players are using the same strategy w. In case of symmetric profiles, if the window size w is w>FS, then, for each flow, a number of packets equal to the FS will be transmitted, while for the remaining (w–FS) dropped packets there will be a reduction of the payoff proportional to the cost factor g. More precisely, the payoff of each flow will be:

$$\text{payoff} = \text{FS} - (\text{w}-\text{FS})\cdot \text{g} = \text{FS}\cdot(g+1)-w\cdot g$$

In the cases where w≤FS, the payoff is equal to the window size, so payoff=w.

If we analyze every scenario of this game, we notice that the payoff function (PF) of each flow is a piecewise linear function of w. The following analysis illustrates how to determine the PF for the above example with N=5 and C=60.

1. For window sizes w, such that w≤12:

There are 1,2,3,4 or 5 active players with probability 1/5 for each case. In all these cases, the window size is smaller than the equivalent FS. So in any of these cases payoff=w. Thus, the expected payoff P(w) is:

$$P(w)=\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}w=w$$

2. For window sizes w, such that 12<w≤15:

In the case where all 5 players are active, then w>FS=12 and

$$\text{payoff} = \text{FS}\cdot(g+1)-w\cdot g = 12\cdot(g+1)-w\cdot g$$

In the remaining cases with 4, 3, 2 and 1 active player(s), payoff=w, because the equivalent FS of each case is greater than any possible value of w in (12, 15]. Thus:

$$P(w)=\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}[12\,(g+1)-wg] \Rightarrow$$
$$P(w)=\frac{4}{5}w+\frac{12}{5}(g+1)-\frac{1}{5}wg=w\left(\frac{4-g}{5}\right)+\frac{12}{5}(g+1)$$

Working in a similar way, we obtain that:

3. For 15<w≤20 the payoff is:

$$P(w)=\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}[15\ (g+1)-wg]+\frac{1}{5}[12\ (g+1)-wg]=>$$

$$P(w)=\frac{3}{5}w+\frac{(12+15)}{5}(g+1)-\frac{2}{5}wg=w\left(\frac{3-2g}{5}\right)+\frac{27}{5}(g+1)$$

4. For 20<w≤30 the payoff is:

$$P(w)=\frac{1}{5}w+\frac{1}{5}w+\frac{1}{5}[20(g+1)-wg]+\frac{1}{5}[15(g+1)-wg]+\frac{1}{5}[12(g+1)-wg]=>$$

$$P(w)=\frac{2}{5}w+\frac{(12+15+20)}{5}(g+1)-\frac{3}{5}wg=w\left(\frac{2-3g}{5}\right)+\frac{47}{5}(g+1)$$

5.For 30<w≤60 the payoff is:

$$P(w)=\frac{1}{5}w+\frac{1}{5}[30(g+1)-wg]+\frac{1}{5}[20(g+1)-wg]$$

$$+\frac{1}{5}[15(g+1)-wg]+\frac{1}{5}[12(g+1)-wg]=>$$

$$P(w)=\frac{1}{5}w+\frac{(12+15+20+30)}{5}(g+1)-\frac{4}{5}wg=w\left(\frac{1-4g}{5}\right)+\frac{77}{5}(g+1)$$

In the above analysis we can observe how the PF changes with respect to the intervals that are defined by the possible FS values. By generalizing the above analysis to any number of players N, we obtain that:

$$P(w)=w\left(\frac{X-Yg}{N}\right)+\frac{Z}{N}(g+1)\ ,\ \forall w>\frac{C}{N}$$
$$P(w)=w\ ,\ \forall w\leq\frac{C}{N} \qquad (2)$$

where X,Y and Z are variables, whose values depend on the parameters of the game instance.

**Theorem 1: In a MaxMin game with capacity C and cost g, and a random number n of active players, where n is uniformly distributed in 1,2,.., or N, the payoff function (PF) for symmetric profiles is:**

$$P(w)=\frac{1}{N}\left[w(a-(N-a)g)+(g+1)\sum_{i=1}^{N-a}\left(\frac{C}{N+1-i}\right)\right],$$
$$\forall a=1,2...N-1\ ,\ \frac{C}{a+1}<w\leq\frac{C}{a}$$
$$P(w)=w\ ,\ \forall w\leq\frac{C}{N}$$

**Proof:** We will derive closed form expressions for the terms X, Y and Z of Equation 2. The term X is the number of the "w/N" terms, which is simply the number of the cases (game states) where w<FS. On the other hand, Y is the number of "-wg/N" terms, which corresponds to the number of the cases where w≥FS. So, we have shown that X+Y=N.

For N players, the possible FS values define a set of N-1 consecutive intervals. For α=1,2,..,N-1, let $s_\alpha$ be the interval between the FS's for n=α+1 and n=α active players. That is, $s_\alpha$ is the following interval:

$$\frac{C}{a+1}<w\leq\frac{C}{a}$$

Note, that α gives the number of possible FS values (number of the game-states) where w≤FS at the corresponding $s_\alpha$ interval. That is because for $w\in s_\alpha$,

$$w\leq\frac{C}{a}<\frac{C}{a-1}<\frac{C}{a-2}<\ldots<\frac{C}{1}$$

As noted earlier, the number of cases where w≤FS is equal to X. Thus, X=α. From X+Y=N we conclude Y=(N-α). In the analysis of the PF for N=5 and C=60, as well as for any N and C, we observe that Z is the coefficient of the term "(g+1)/N" at every interval for w, defined by the possible FS values. The value of Z is equal to the sum of all the FS's whose value is lower than the possible values of w at the specific interval. For example, for N=5, C=60 and 30<w≤60, the coefficient of "(g+1)/N" is (12+15+20+30), which corresponds to the sum

$$\frac{C}{N}+\frac{C}{N-1}+\ldots+\frac{C}{a+1}$$

In general, variable Z is equal to the sum

$$Z=\sum_{i=1}^{N-a}\left(\frac{C}{N+1-i}\right)$$

**Theorem 2: For α=1,2,..,N-1, let $g_\alpha$ be**

$$g_a=\frac{a}{N-a}$$

**Then, the symmetric profile w=C/α gives the best possible payoff, if**

| | |
|---|---|
| **$0\leq g\leq g_1$,** | **when α=1,** |
| **$g_{\alpha-1}\leq g\leq g_\alpha$,** | **when α=2,3,…,N-1.** |

**If g is greater than $g_{N-1}$, then w=C/N gives the best possible payoff. There is an optimal strategy w for each such interval of g. Note that when g has one of the boundary values $g_1$, $g_2$,…,$g_{N-1}$ it belongs to two intervals in the above definition. For boundary values of g, all values of w in the closed interval defined by the two optimal values for w, give the best possible payoff.**

**Proof:** Differentiating P(w) with respect to w (for each interval $s_\alpha$, since P(w) is a piecewise function) of Theorem 1 and setting it equal to zero we obtain:

$$\frac{dP(w)}{dw}=0\ =>\ a-(N-a)g=0\ =>g=\frac{a}{N-a}$$

By solving the above equation for each of the $s_\alpha$ intervals, we can find the value of g for which the slope of P(w) is zero. Low values of the cost factor g provide incentives for the flow to play aggressively. Consequently, for very low values of g, starting from zero, the slope of P(w) is positive for all $w<C$. For larger values of g, strategies with large window sizes will suffer larger penalties for dropped packets. There is a boundary value of g ($dP(w)/dw=0$) at which P(w) remains the same for all values of $w \in s_1$. Any greater value of g will make the slope of P(w) at $s_1$ negative.

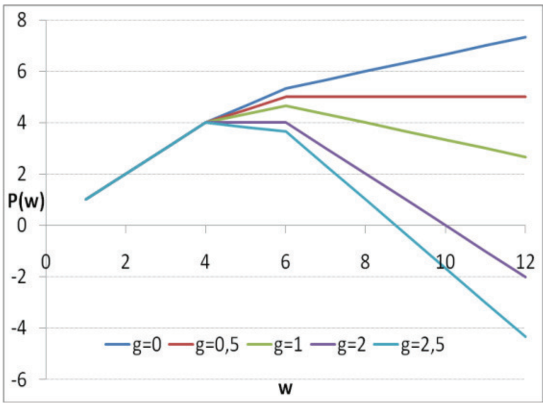


**Figure 1. P(w) for several g values (C=12 ,N=3).**

In Figure 1, each curve represents the function P(w) for a specific value of g. The curve (g=0) gives the best P(w) for every w among all curves, since it incurs no penalty. For all other curves, we observe that the slopes gradually decrease as the value of w increases, and in some curves the slope becomes negative (e.g. g=1, for w in {6,7,…,12}).

The curves of Figure 1 reveal a simple procedure to find the optimal symmetric profile for each corresponding value of g. Given a specific value for g, we can start from interval $s_1$, which involves the greatest values of w, and check one-by-one the slope of all the intervals, in search of the interval with the largest values of w with non-negative slope. The upper bound of this interval will give the **best possible payoff.**

In the special cases where the derivate equals zero, all values of w in that interval achieve the best possible payoff. In the rest of this work, when the best payoff is achieved by an interval of values, we will simply work with the upper bound of the interval. The upper bound is always one of the possible FS values.

**Theorem 3: The symmetric profiles proposed by Theorem 2 are symmetric NE of the game.**

**Proof:** To prove that these strategy sets are NE, we will show that no player has an incentive to unilaterally differentiate her strategy. Let $w_k$ be the common strategy with the best payoff and player A the only player who changes her strategy from $w_k$ to $w_A \neq w_k$.

**1. Let $w_A<w_k$.**

Let $FS_i$ be the FS when i players are active. Assume i such that:

$$FS_{i+1} < w_A \leq FS_i$$

Player A will receive a payoff equal to:

- $\mathbf{w_A}$ , for each state with j=1,2,…,i active players
- $\mathbf{[FS_j - (w_A - FS_j) \cdot g]}$ , for each state with j=i+1,..,N active players.

This is because the transmission of the window size is guaranteed if the window size does not exceed the FS of the state. If $w_A$ exceeds the FS of the state, since the common strategy $w_k$ is $w_A<w_k$, all players play above the FS, therefore they all receive the FS (of the state) minus the cost of their extra packets which are dropped.

The way we calculate the expected payoff of player A as a weighted average of the payoff of each possible state is given as the sum of exactly the same terms that are summed in the general equation of Theorems 1. Thus, when player A unilaterally differentiates her strategy from $w_k$ to $w_A$, where $w_A<w_k$, she receives the same payoff as if all players had chosen as common strategy the symmetric profile with window size $w_A$. In Theorem 2 we proved that the symmetric profiles that are proposed give the best possible payoff among all symmetric profiles.

Consequently, for one player who unilaterally differentiates her strategy from the symmetric profile $w_k$ to a strategy $w_A$ it holds:

$$\mathrm{P}(w_A) \leq P(w_k), \forall w_A < w_k$$

**2. Let $w_A>w_k$.**

Let k be the number of players that corresponds to the fair share FS that is equal to the best symmetric strategy $w_k$ ($FS_k=C/k=w_k$), as given by Theorem 2. Let $w_A=w_k+x$. We partition the field of possible values of $w_A$ into intervals.

2.1. Let $w_A \in (w_k, 2w_k]$.

Then $x \in (0, w_k]$. Player A will receive a payoff equal to:

- $\mathbf{w_A}$, for each state with j=1,2,…,k-1 active players.
- $\mathbf{[FS_j - (w_A - FS_j) \cdot g]}$, for each state with j=k, k+1,..,N active players.

Among the states with 1,2,…,k-1 active players, the greatest congestion occurs at the state where k-1 players are active. At this worst case, all the other k-2 players choose $FS_k$, so the remaining capacity for player A is:

$$2 \cdot w_k \geq w_A = w_k + x$$

Thus, at these states player A has a successful transmission of her window size. On the other hand, when k players are active and all the other k-1 players chose $FS_k$, the remaining capacity for player A is $FS_k$. Similarly, for all other states with j=k+1,..,N active players, player A can send successfully the $FS_j$ of each state and will suffer the cost of her $w_A$-$FS_j$ dropped packets.

The terms that we add to find the expected payoff of player A are equal to the terms that we need to add to calculate the expected payoff of symmetric profiles at the interval:

$$\frac{C}{k} < w \leq \frac{C}{k-1}$$

If all players chose a symmetric profile in the interval $(FS_k,FS_{k-1}]$, then they would successfully transmit their window size at the states where j=1,2,…,k-1 players were active, while at all other states they would successfully transmit the FS of each state. All symmetric profiles at this interval will give a lower payoff than the symmetric profile $w_k$, as shown in Theorem 2, and the linear function would be decreasing in this interval. Since the payoff for player A with strategy $w_A \in (w_k,2w_k]$ when all other players stay at $w_k$ is given by the same decreasing linear function that gives the payoff of the symmetric profiles of the interval $(FS_k,FS_{k-1}]$, we conclude that:

When one player A unilaterally differentiates her strategy from the symmetric profile $w_k$ to a strategy $w_A$, there is no $w_A \in (w_k,2w_k]$, such that $P(w_A)>P(w_k)$.

Comparing, at each possible state, player's A payoff when she chooses $w_A$ to the payoff she receives when she stays at $w_k$ (let $p_{ki}$ be this payoff, where i is the number of active players of the state), given that all other players play $w_k$, we notice that with $w_A$ player A receives:

- $\mathbf{p_{ki}+x}$, for all states with i=1,2,..,k-1 active players
- $\mathbf{p_{ki}-x \cdot g}$, for all states with i=k,k+1,…N active players

Since $P(w_A) \leq P(w_k)$,

$$P(w_A)-P(w_k)=[x(k-1)/N]-[x \cdot g(N-k+1)/N] \leq 0. \qquad (3)$$

2.2 Let $w_A \in (h \cdot w_k,(h+1)w_k]$, $\forall h \in (2,3,...,k-1]$.

At these k-2 intervals, we compare again player's A payoff when she chooses $w_k$ or $w_A$, given that all other players play $w_k$. We notice that with $w_A$ player A receives:

- $\mathbf{p_{ki} + x}$, for all states with i=1,2,..,k-h active players, because even when the greatest congestion occurs at the state where k-h players are active, all the other k-h-1 players choose $FS_k$, so the remaining capacity for player A is: $(h+1) \cdot w_k \geq w_A = w_k + x$
- $\mathbf{p_{ki} + [C - w_k \cdot (i-1)] < p_{ki} + x}$, but with an extra penalty of $\mathbf{g \cdot [w_A-[C-w_k \cdot (i-1)]]>0}$ for all states with i=k-h+1,…,k-1 active players. Let $\mathbf{Q=g \cdot [w_A-[C-w_k \cdot (i-1)]]}$ be this penalty. Let Y such that $\mathbf{[C-w_k \cdot (i-1)]+Y=x.}$ Then we will assume that she receives $\mathbf{p_{ki}+x}$ with an **extra penalty of Z=Q+Y.** (We add and subtract Y).
- $\mathbf{p_{ki}-x \cdot g}$, for all states with i=k,k+1,…N active players. In these states all players receive the $FS_i$ of each state minus the cost of their dropped packets. Strategy $w_A$ loses x more packets than strategy $w_k$.

So, for $w_A \in (h \cdot w_k,(h+1)w_k]$, there is:

- a linear increase for each extra packet, similar to the increase when $w_A \in (w_k,2w_k]$, for all states with i=1,2,..,k-h active players;
- a linear increase of the penalty for the dropped packets for each extra packet, for all states with i=k,k+1,…N active players;
- an extra penalty, previously named as Z, for all states with i=k+h,…,k-1 active players.

So, comparing the payoff of player A for $w_A$ and $w_k$, given that all other players play $w_k$:

$$P(w_A)-P(w_k)=[x(k-1)/N]-[x \cdot g(N-k+1)/N]-Z, \quad \text{where } Z>0. \qquad (4)$$

The difference of the first two terms $[x(k-1)/N]-[x \cdot g(N-k+1)/N]$ does not exceed zero, as shown in Equation 3. Since we subtract a non-negative value Z from a non-positive difference, we conclude that in Equation 4:

$$P(w_A)-P(w_k) \leq 0.$$

So, there is no $w_A \in (h \cdot w_k,(h+1)w_k]$, $\forall h \in (2,3,...,k-1]$, such that $P(w_A)>P(w_k)$.

Since there is no possible deviation that can offer an improved payoff to any player who unilaterally changes her strategy, **this strategy set (or else strategy profile) is a NE**.

## 4. CONCLUSION

We examined Bayesian Window-games on a MaxMin router and showed that the MaxMin queue policy leads to fair NE. The strategy profiles of the possible NE depend on the value of g. For plausible values of g (for example g=1), the NE strategy profiles utilize a sufficiently large part of the router capacity, while being at the same time absolutely fair.

Finally, we believe that the NE described in Theorem 3 are the only symmetric NE of the game. In particular, we believe that we can use the machinery of Theorems 1, 2 and 3 to show this result and intend to derive a proof in our future work.

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